

January 19, 2012
TAUP-2941/11

Holographic realization of large- N_c orbifold equivalence with non-zero chemical potential

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Abstract

Recently, it has been suggested that large- N_c orbifold equivalences may be applicable to certain theories with chemical potentials, including QCD, in certain portions of their phase diagram. When valid, such an equivalence offers the possibility of relating large- N_c QCD at non-zero baryon chemical potential, a theory with a complex fermion determinant, to a related theory whose fermion determinant is real and positive. In this paper, we provide a test of this large N_c equivalence using a holographic realization of a supersymmetric theory with baryon chemical potential and a related theory with isospin chemical potential. We show that the two strongly-coupled, large- N_c theories are equivalent in a large region of the phase diagram.

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Contents

1	Introduction	2
2	Orbifold projections for large-N_c QCD	5
2.1	From $SO(2N_c)_F$ to QCD with a baryon chemical potential	6
2.2	From $SO(2N_c)_F$ to QCD with an isospin chemical potential	7
2.3	Validity of large- N_c equivalences and their application to the sign problem	8
3	A holographic realization	9
3.1	Orientifold and orbifold projections	9
3.2	From isospin to baryon chemical potential	11
3.3	Validity of the equivalence	12
3.4	N_f/N_c corrections	14
4	Conclusion	15
A	A quick introduction to the sign problem	16
B	Projection of the DBI action	17

1 Introduction

Understanding QCD at non-zero baryon density is an important goal, both for its intrinsic interest and for applications such as the structure of neutron stars and the mechanism of core-collapse supernova. Due to the notorious *sign problem*,⁵ we lack generally effective methods for performing numerical simulations of gauge theories with a baryon chemical potential. When a non-zero baryon number chemical potential is present, the determinant of the Euclidean Dirac operator is no longer positive and standard Markov-chain Monte-Carlo methods are not applicable. Although many schemes have been proposed to address the sign problem [1, 2, 3, 4, 5, 6, 7], it is fair to say that no fully satisfactory solution has been found. At the same time, condensed matter phases of several QCD-like theories which do not suffer from the sign problem have been studied numerically, in the hope that one may extract lessons about strongly interacting finite density systems which will also apply to QCD at finite baryon density. Examples include $SU(2)$ Yang-Mills (YM) with even numbers of fundamental flavors [8, 9], $SU(N_c)$ YM with adjoint fermions [9], and QCD with an isospin chemical potential [10, 11]. However, there is no solid argument delineating the extent to which these theories can reproduce properties of QCD with a baryon chemical potential.

In recent years, it has been understood that a network of large- N_c equivalences relate various non-Abelian gauge theories with differing gauge groups and matter content [12, 13, 14, 15, 16]. These equivalences, which are generated by appropriate orbifold projections, relate the leading large N_c behavior of connected correlators of specific classes of observables. The large- N_c equivalences are valid provided certain symmetry realizations are satisfied [17]. For example, $SU(N_c)$

⁵More properly, this should be called a *phase* problem. See Appendix A for a brief summary.

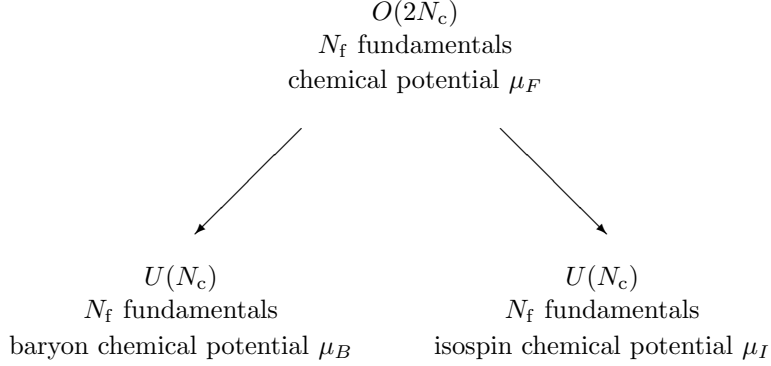


Figure 1: An orbifold projection acting on a parent $O(2N_c)$ Yang-Mills theory with N_f fundamental representation Dirac fermions and a flavor-singlet chemical potential μ_F may generate a $U(N_c)$ daughter theory with N_f fermions and a baryon chemical potential μ_B (left) or, provided N_f is even, the same $U(N_c)$ theory with an isospin chemical potential μ_I (right) [21]. In the parent $O(2N_c)$ theory with even N_f , there is no distinction between a baryon or isospin chemical potential.

and $SO(2N_c)$ Yang-Mills theories have coinciding large N_c limits of all Wilson loop expectation values (as well as connected correlators of real parts of Wilson loops), provided charge conjugation symmetry is not spontaneously broken in the $SU(N_c)$ theory [18].

We will be concerned with QCD-like theories containing fundamental representation fermions and non-zero chemical potentials. Specifically, we will discuss:

1. $SO(2N_c)$ Yang-Mills with N_f Dirac fundamental representation fermions and a non-zero fermion chemical potential μ_F , under which all N_f flavors have charge +1. For brevity, we will denote this theory as $SO(2N_c)_F$.
2. $SU(N_c)$ Yang-Mills with N_f fundamental representation fermions and a non-zero baryon chemical potential μ_B , under which all N_f fermion flavors have charge +1. For brevity, we will denote this theory as QCD_B .
3. $SU(N_c)$ Yang-Mills with N_f fundamental representation fermions, with N_f even, and a non-zero isospin chemical potential μ_I , under which half the fermion flavors have charge +1 and half have charge -1 . For brevity, we will denote this theory as QCD_I .

Although QCD_B suffers from a sign problem, this is not the case for either QCD_I or $SO(2N_c)_F$, as both of these theories have a real and positive fermion determinant [10, 21].

As figure 1 schematically depicts, starting from the $SO(2N_c)_F$ theory one choice of orbifold projection yields QCD_B , while a different choice yields QCD_I .⁶ Based on this observation, it has recently been suggested that large N_c equivalences may relate suitable observables in the parent

⁶ Strictly speaking, the orbifold projection maps a parent theory with $SO(2N_c)$ gauge group to a daughter $U(N_c)$ gauge theory. But the difference between $U(N_c)$ and $SU(N_c)$ theories is sub-dominant in the large N_c limit. Note that in a $U(N_c)$ theory with chemical potential, the $U(1)$ part of the gauge field is to be fixed at infinity. In finite volume, the theory should be defined with Dirichlet boundary conditions on the $U(1)$ gauge field, not periodic. (In practice, for lattice simulations, it is more convenient to simply use the $SU(N_c)$ theory.)

$SO(2N_c)_F$ theory to corresponding observables in either QCD_B or QCD_I [21, 22, 23]. In portions of the phase diagram where both equivalences are valid (if such regions exist), this implies that one may obtain quantitative information about large- N_c QCD with a baryon chemical potential from studies of the same theory with an isospin chemical potential, thereby circumventing the sign problem.⁷

When $N_c \rightarrow \infty$ with N_f fixed, a comparison of planar Feynman diagrams in the parent $SO(2N_c)$ and daughter $SU(N_c)$ theories shows that they coincide [12, 13, 21, 22]. (For other approaches see [15].) This analysis is unaffected by the presence of a non-zero chemical potential. Coinciding perturbative expansions does not, however, necessarily imply a genuine non-perturbative equivalence. Necessary conditions for a valid equivalence include a requirement that the symmetries used to define the orbifold projection not be spontaneously broken in the parent theory [15]. Since the projection leading to QCD_B is generated by a combination of a gauge transformation and a $U(1)_F$ phase rotation, this projection can only lead to a valid large- N_c equivalence in portions of the phase diagram where the $U(1)_F$ global symmetry associated with net fermion number is unbroken. In other words, a large N_c equivalence relating QCD with baryon and isospin chemical potentials can only apply to portions of the phase diagram in which fermions do not condense to form a superfluid. In simpler examples, analogous conditions on symmetry realizations are both necessary and sufficient conditions for the validity of large N_c equivalences [15]; whether this is the case in the present example is not yet clear.

In this paper, we use gauge/gravity duality to test the validity of analogous possible large N_c equivalences relating supersymmetric generalizations of the above theories. Although the trio of theories we consider will not include QCD itself, the arguments of refs. [21, 22, 23] are equally applicable to the supersymmetric theories we consider. By considering supersymmetric theories, and using holographic methods, it will be possible to examine relations between theories with different types of chemical potential directly in the limit of strong coupling (and large N_c) using simple analytic methods. We will find that, in a large region of the phase diagram with no spontaneous breaking of flavor symmetries, large N_c equivalences between our theories are valid.

Our holographic construction involves an orbifold and an orientifold projection of the D3/D7-system, with N_c D3 branes, N_f D7 branes, and always $N_f \ll N_c$. At large- N_c and strong 't Hooft coupling, the low energy theory on the D3 branes is described by classical type IIB supergravity in $AdS_5 \times S^5$, with probe D7 branes wrapping an S^3 in the S^5 [24]. The projections act on the geometry, changing it to $AdS_5 \times \mathbf{RP}^5$. An isospin chemical potential in the original theory is described by a particular configuration of the gauge field on the D7 brane. We show that after the projection the isospin chemical potential becomes a baryon chemical potential. Figure 2 illustrates the connections between the corresponding field theories. We prove that, provided the projection symmetries are not broken, the equations of motion of the D7 branes coincide in both theories. As we discuss, this implies that the conjectured large N_c equivalences are valid in these theories in those regions of the phase diagram where flavor symmetries are unbroken and no additional fields become active.

The paper is organized as follows: in § 2 we review in more detail the proposed equivalences involving large- N_c QCD. In § 3 we provide a holographic realization for our supersymmetric generalization. Finally, in § 4 we discuss the regime of validity of the equivalence, and the consequences for the phase diagram.

⁷Readers should refer to section 2, and refs. [21, 22, 23], for more details and more nuanced discussion.

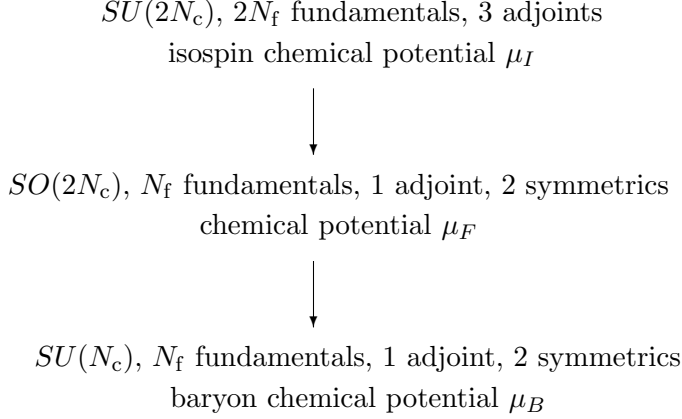


Figure 2: $\mathcal{N} = 1$ supersymmetric theories related by orbifold projections. In the $SO(2N_c)$ theory, there is no distinction between a baryon or isospin chemical potential.

2 Orbifold projections for large- N_c QCD

Consider an $SO(2N_c)$ Yang-Mills theory coupled to N_f fundamental representation Dirac fermions. The Lagrange density is

$$\mathcal{L}_{SO} = \frac{1}{4g_{SO}^2} \text{tr} F_{\mu\nu}^2 + \sum_{a=1}^{N_f} \bar{\psi}_a (\gamma^\mu D_\mu + m_q + \mu_F \gamma^0) \psi_a, \quad (1)$$

where $F_{\mu\nu}$ is the field strength of the $SO(2N_c)$ gauge field A_μ , $D_\mu \equiv \partial_\mu + A_\mu$, ψ_a is a Dirac fermion in the vector representation of $SO(2N_c)$, and m_q and μ_F are the quark mass and fermion chemical potential, respectively. Because the gauge field is real, the Dirac operator $\mathcal{D} \equiv (\gamma^\mu D_\mu + m_q + \mu_F \gamma^0)$ satisfies $(C\gamma_5)D(C\gamma_5)^{-1} = D^*$, where C is the charge conjugation matrix defined by $C\gamma_\mu C^{-1} = -\gamma_\mu^T = -\gamma_\mu^*$.⁸ If v is an eigenvector of the Dirac operator \mathcal{D} with an eigenvalue λ , $\mathcal{D}v = \lambda v$, then $(C\gamma_5)^{-1}v^*$ is another eigenvector of \mathcal{D} with eigenvalue λ^* , and is linearly independent of v even when λ is real. (See § 2.3 of ref. [20].) Therefore the determinant of D is always real and positive, implying that standard Markov chain Monte-Carlo simulation techniques may be used [21].

When $m_q = \mu_F = 0$, the Lagrangian (1) has a manifest $SU(N_f)_L \times SU(N_f)_R \times U(1)_F \times U(1)_A$ flavor symmetry, just like $SU(N_c)$ QCD. However, the flavor symmetry of the theory is actually larger than this due to the fact that $SO(2N_c)$ is a real gauge group; classically it extends to $U(2N_f)$ [25, 26]. The axial $U(1)_A \subset U(2N_f)$ is anomalous, and at the quantum level the (continuous part of the) flavor symmetry is $SU(2N_f)$, which spontaneously breaks to $SO(2N_f) \supseteq SU(N_f)_V$ due to the formation of a chiral condensate $\langle \bar{\psi}\psi \rangle$. The resulting massless Nambu-Goldstone bosons span the $SU(2N_f)/SO(2N_f)$ coset space. In contrast to QCD, some of these Nambu-Goldstone bosons, which we will refer to as baryonic pions, are charged under $U(1)_F$. Ordinary pions are created by operators that look like $\bar{\psi}_a \gamma_5 \psi_b$, while baryonic pions are created by color-singlet operators of the form $\psi_a^T C \gamma_5 \psi_b$ and $\bar{\psi}_a C \gamma_5 \bar{\psi}_b^T$.

⁸We use (++++) metric signature and Hermitian gamma matrices.

2.1 From $SO(2N_c)_F$ to QCD with a baryon chemical potential

To perform an orbifold projection, one identifies a discrete subgroup of the symmetry group of the parent theory, which for us is the $SO(2N_c)_F$ theory, and then removes all of the degrees of freedom in the parent theory which are not invariant under the chosen discrete symmetry. This yields a daughter theory, which will turn out to be large- N_c QCD.

The required orbifold projection is a \mathbb{Z}_2 subgroup of the $SO(2N_c)$ gauge $\times U(1)_F$ flavor symmetry of the $SO(2N_c)_F$ theory. To define the orbifold projection, which we will denote as \mathcal{P}_B , take $J_{2N_c} \in SO(2N_c)$ to be given by $J_{2N_c} = i\sigma_2 \otimes \mathbf{1}_{N_c}$, where $\mathbf{1}_N$ denotes an $N \times N$ identity matrix. The group element J_{2N_c} generates a \mathbb{Z}_4 subgroup of $SO(2N_c)$. Next, let $\omega = e^{i\pi/2} \in U(1)_F$ denote the phase which generates a \mathbb{Z}_4 subgroup of $U(1)_F$. The discrete symmetry which will define the orbifold projection acts on the fields A_μ , ψ_a as

$$A_\mu \rightarrow J_{2N_c} A_\mu J_{2N_c}^{-1}, \quad \psi_a \rightarrow \omega J_{2N_c} \psi_a. \quad (2)$$

Since $J_{2N_c}^2 = -\mathbf{1}_{2N_c}$, and $(\omega J_{2N_c})^2 = +\mathbf{1}_{2N_c}$, this symmetry transformation generates a \mathbb{Z}_2 subgroup of $SO(2N_c) \times U(1)_F$.

The action of the orbifold projection on the basic fields is

$$\mathcal{P}_B A_\mu = \frac{1}{2} (A_\mu + J_{2N_c} A_\mu J_{2N_c}^{-1}), \quad \mathcal{P}_B \psi = \frac{1}{2} (\psi + J_{2N_c} \psi K_{N_f}^{-1}) \quad (3)$$

where, for later convenience, we have defined a matrix $K_{N_f}^{-1} \equiv i\mathbf{1}_{N_f}$ acting on flavor indices. To display the action of the projection more explicitly, it is convenient to block-decompose the gauge and fermion fields. The gauge field A_μ may be written in terms of four $N_c \times N_c$ blocks as

$$A_\mu \equiv \begin{pmatrix} A_\mu^A + B_\mu^A & C_\mu^A - D_\mu^S \\ C_\mu^A + D_\mu^S & A_\mu^A - B_\mu^A \end{pmatrix}, \quad (4)$$

where fields marked with an ‘A’ or ‘S’ superscript are anti-symmetric or symmetric matrices, respectively. Under the \mathbb{Z}_2 symmetry transformation (3), A_μ^A , and D_μ^S are even while B_μ^A , and C_μ^A are odd, so the orbifold projection sets $B_\mu^A = C_\mu^A = 0$. Hence

$$\mathcal{P}_B A_\mu = \begin{pmatrix} A_\mu^A & -D_\mu^S \\ D_\mu^S & A_\mu^A \end{pmatrix}. \quad (5)$$

If one defines the unitary matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{N_c} & i\mathbf{1}_{N_c} \\ \mathbf{1}_{N_c} & -i\mathbf{1}_{N_c} \end{pmatrix}, \quad (6)$$

then

$$P \mathcal{P}_B A_\mu P^{-1} = \begin{pmatrix} \mathcal{A}_\mu & 0 \\ 0 & -\mathcal{A}_\mu^T \end{pmatrix}, \quad (7)$$

where $\mathcal{A}_\mu \equiv A_\mu^A + iD_\mu^S$ is a $U(N_c)$ gauge field. At large N_c , we can neglect the difference between $U(N_c)$ and $SU(N_c)$ up to $1/N_c^2$ corrections.

We can split the $2N_c$ -component fundamental fermions of the $SO(2N_c)$ theory into two N_c -component fields,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (8)$$

and then we use the matrix (6) to change basis. This yields

$$P\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (9)$$

where $\psi_{\pm} \equiv (\psi_1 \pm i\psi_2)/\sqrt{2}$. From eq. (7), one sees that ψ_+ and ψ_- transform as fundamental and antifundamental representations under $SU(N_c)$, respectively. After the projection, only ψ_+ survives.

If we take the Lagrangian of the parent theory and apply the orbifold projection, it becomes

$$\mathcal{L} = \frac{1}{4g_{SU}^2} \text{Tr} \mathcal{F}_{\mu\nu}^2 + \sum_{a=1}^{N_f} \bar{\lambda}_a (\gamma^\mu \mathcal{D}_\mu + m_q + \mu_B \gamma^4) \lambda^a, \quad (10)$$

where $\mathcal{F}_{\mu\nu}$ is the field strength of the $SU(N_c)$ gauge field $\mathcal{A}_\mu = A_\mu^A + iD_\mu^S$, $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$, $\lambda^a = \sqrt{2}\psi_+^a$, and the gauge coupling is given by $g_{SU}^2 = g_{SO}^2$.

In the large- N_c limit for fixed N_f , connected correlation functions of operators $\mathcal{O}_i^{(p)}$ in the parent SO theory which are invariant under the projection symmetry, and their counterparts $\mathcal{O}_i^{(d)}$ in the daughter SU theory which are formed from the projected fields, coincide to all orders in perturbation theory [12],

$$\langle \mathcal{O}_1^{(p)} \mathcal{O}_2^{(p)} \dots \rangle_p = \langle \mathcal{O}_1^{(d)} \mathcal{O}_2^{(d)} \dots \rangle_d. \quad (11)$$

The baryonic pion fields do not survive the projection, so there is no equivalent to them in the daughter theory.

2.2 From $SO(2N_c)_F$ to QCD with an isospin chemical potential

When the number of flavors in the parent $SO(2N_c)$ theory is even, $N_f = 2k$, it is also possible to define a projection which yields large- N_c QCD with an isospin chemical potential. The projection for the gauge field is the same as in eq. (3), but we now choose a different orbifold action on the flavor indices of the fermions. Let us write the fermions using $N_c \times N_f$ -component fields as

$$\psi = \begin{pmatrix} \psi_+^{(1)} & \psi_+^{(2)} \\ \psi_-^{(1)} & \psi_-^{(2)} \end{pmatrix}. \quad (12)$$

In this basis, the orbifold action is

$$\psi \rightarrow J_{2N_c} \psi J_{2k}^{-1}. \quad (13)$$

This transformation also generates a \mathbb{Z}_2 group. The action of the orbifold projection \mathcal{P}_I is

$$\mathcal{P}_I A_\mu = \frac{1}{2} (A_\mu + J_{2N_c} A_\mu J_{2N_c}^{-1}), \quad \mathcal{P}_I \psi = \frac{1}{2} (\psi + J_{2N_c} \psi J_{2k}^{-1}). \quad (14)$$

Defining $\varphi_{\pm} = (\psi_{\pm}^{(1)} \mp i\psi_{\pm}^{(2)})/\sqrt{2}$ and $\xi_{\pm} = (\psi_{\pm}^{(1)} \pm i\psi_{\pm}^{(2)})/\sqrt{2}$, one sees that φ_{\pm} survive while ξ_{\pm} is eliminated by the projection (14). Since φ_+ and φ_- couple to \mathcal{A}_{μ} and \mathcal{A}_{μ}^C , respectively, the fermionic part of the action of the daughter theory can be written as

$$\sum_{f=1}^k \sum_{\pm} \bar{\lambda}_{\pm}^{(f)} (\gamma^{\mu} D_{\mu} + m \pm \mu \gamma^4) \lambda_{\pm}^{(f)}, \quad (15)$$

where $\lambda_{+}^{(f)} = \sqrt{2} \varphi_{+}^{(f)}$, $\lambda_{-}^{(f)} = \sqrt{2} (\varphi_{-}^{(f)})^C$, and we have now written the flavor index $(f) = 1, \dots, k$ explicitly. This theory has an isospin chemical potential $\mu_I \equiv 2\mu$.

2.3 Validity of large- N_c equivalences and their application to the sign problem

The perturbative proof of the parent-daughter equivalence with isospin chemical potential is valid also when quark loops are included in planar diagrams, so it is possible to extend the analysis to include N_f/N_c corrections. However, this is not possible for the projection to a theory with baryon chemical potential. The difference stems from the properties of the projection in the flavor sector, while the projection to isospin chemical potential is performed using a *regular* representation [12, 13]

$$\text{tr } J_{2k} = 0, \quad J_{2k}^2 = \pm \mathbf{1}_{2k}, \quad (16)$$

these conditions are not satisfied for the representation used to do the projection to the theory with a baryon chemical potential, where we have used K_{2k} instead of J_{2k} . Only diagrams containing a single quark loop produce the same result in parent and daughter theories.

To go beyond the perturbative proof of the equivalence one needs to do a careful analysis of the necessary and sufficient conditions that must be obeyed for the equivalence to hold. A necessary condition is that the projection symmetry not be spontaneously broken in the parent [15]. The $U(1)_B$ symmetry, which is used for the projection from the $SO(2N_c)$ theory to QCD with a baryon chemical potential, breaks to \mathbb{Z}_2 when the baryonic pion condenses (*e.g.*, when $\mu > m_{\pi}/2$ at zero temperature). Therefore, the parent-daughter equivalence can hold only at smaller values of the chemical potential.⁹ On the other hand, the projection symmetry to obtain QCD with isospin chemical potential should not be spontaneously broken for any μ ; in this case condensation of baryonic pions in the parent theory is mapped to pion condensation in the daughter theory.

Clearly, if it were possible to show that these equivalences hold nonperturbatively, they would be very useful because one would be able to derive properties of a large- N_c QCD theory with baryonic chemical potential from a $SO(2N_c)$ theory or from large- N_c QCD with isospin chemical potential, both of which are free of the sign problem. This could also explain why the phase quenching approximation in QCD is quite good — for a certain class of operators (*e.g.*, the chiral condensate), the phase quenching approximation becomes exact in the large- N_c limit.¹⁰ The phase quenching approximation for the chiral condensate is exact in the chiral random matrix model [39, 22]. The orbifold equivalence, if true, would ensure that the phase quenching approximation in QCD is exact for a large class of observables in the large- N_c limit, even beyond the parameter region where the chiral random matrix model is valid (the “ ϵ -regime”).

⁹Note that the chemical potential at which baryonic pions condense is temperature dependent, and should increase with increasing temperature.

¹⁰Note that (for N_f even) dropping the phase of the fermion determinant turns the functional integral for QCD_B into that for QCD_I .

To provide a nonperturbative proof of the orbifold equivalence in QCD with chemical potentials is beyond the scope of this paper. However, in the following section we will show that analogous equivalences hold in a class of supersymmetric cousins of QCD which have gravity duals.

3 A holographic realization

It is possible to build a simple supersymmetric model where an isospin chemical potential is projected into a baryon chemical potential. The model is one of the examples mentioned in ref. [27], based on the description of $\mathcal{N} = 2$ theories from D4 branes suspended between NS5 branes [28]. Flavor can be introduced by adding D6 branes. We will start with a configuration whose low energy limit on the T-dual D3 branes is $\mathcal{N} = 4$ $U(2N_c)$ super Yang-Mills plus $2N_f$ hypermultiplets in the fundamental representation, so the flavor group is $U(2N_f)$. In the T-dual configuration the flavor branes are D7's, and we will work in the 't Hooft limit of $N_f/N_c \ll 1$ so we can neglect their backreaction just as in the D3/D7 system of ref. [24]. We then introduce an orientifold plane to produce an $SO(2N_c)$ theory with $USp(2N_f)$ flavor group and then finally do a \mathbb{Z}_2 orbifold projection that reduces it to $U(N_c)$ with $U(N_f)$ flavor group. We will show that an isospin chemical potential in the original $U(2N_c)$ theory is projected to a baryon chemical potential in the $U(N_c)$ theory and discuss when the two theories are equivalent.

3.1 Orientifold and orbifold projections

The construction in type IIA theory consists on a set of $2N_c$ D4 branes wrapping a circle in the x^6 direction and intersecting two $O6^+$ planes at opposite sides of the circle. In addition, there is a NS5 brane at each orientifold point and $2N_f$ D6 branes parallel to the $O6$ planes:

	0	1	2	3	4	5	6	7	8	9
D4	×	×	×	×	·	·	×	·	·	·
O6/D6	×	×	×	×	·	·	·	×	×	×
NS5	×	×	×	×	×	×	·	·	·	·

Since the $O6$ planes are positively charged, Ramond-Ramond (RR) tadpoles do not cancel and the β function for the 't Hooft coupling is positive. However, in the 't Hooft limit $N_f \ll N_c$, the β function is suppressed by N_f/N_c at large N_c . So to leading order in N_f/N_c we can neglect the tadpoles and consider the $D6$'s and $O6$'s as probes.

This brane setup has as a T-dual a configuration involving D3 and D7 branes. The two $O6$ planes map to a single $O7$ plane and the NS5 brane to a \mathbb{Z}_2 singularity localized at $x^6 = x^7 = x^8 = x^9 = 0$:

	0	1	2	3	4	5	6	7	8	9
D3	×	×	×	×	·	·	·	·	·	·
O7/D7	×	×	×	×	·	·	×	×	×	×
\mathbb{Z}_2	×	×	×	×	×	×	·	·	·	·

The geometric effect of the \mathbb{Z}_2 action is a reflection in the transverse directions. The orientifold projection $\Omega' = \Omega R_{45}(-1)^{F_L}$ involves worldsheet parity reversal Ω , a reflection R_{45} in the x^4 and x^5 coordinates, and $(-1)^{F_L}$ acts as -1 in the Ramond sector of left movers. The effect on Chan-Paton factors of open strings on D3 branes is given by the matrices $\gamma_3 = iJ_{2N_c}$ for the orbifold

action and $\omega_3 = \mathbf{1}_{2N_c}$ for the orientifold action. The corresponding matrices for the D7 branes are $\gamma_7 = iJ_{2N_f}$ and $\omega_7 = iJ_{2N_f}$.

The massless spectrum of D3 branes involves a vector multiplet on the worldvolume A_{0123} and three complex scalar multiplets describing the transverse motion X_{45} , X_{67} , X_{89} . Before the projection those describe the field content of $\mathcal{N} = 4$ $U(2N_c)$ super Yang-Mills, that in $\mathcal{N} = 2$ language involves a vector multiplet and a hypermultiplet in the adjoint representation. The orientifold action is

$$\begin{aligned} A_{0123} &\rightarrow -\omega_3 A_{0123}^T \omega_3^{-1}, \\ X_{45} &\rightarrow -\omega_3 X_{45}^T \omega_3^{-1}, \\ X_{67,89} &\rightarrow \omega_3 X_{67,89}^T \omega_3^{-1}. \end{aligned} \quad (17)$$

Therefore, the orientifold projection for the gauge field is

$$\mathcal{P}_\omega A_\mu = \frac{1}{2} (A_\mu - A_\mu^T), \quad (18)$$

so the projected gauge field is antisymmetric and spans an $SO(2N_c)$ algebra. The field X_{45} is in an antisymmetric (adjoint) representation, while for the fields $X_{67,89}$ the orientifold action projects them to a symmetric representation.

The \mathbb{Z}_2 action of the orbifold is

$$\begin{aligned} \mathcal{P}_\omega A_{0123} &\rightarrow \gamma_3 \mathcal{P}_\omega A_{0123} \gamma_3^{-1}, \\ \mathcal{P}_\omega X_{45} &\rightarrow \gamma_3 \mathcal{P}_\omega X_{45} \gamma_3^{-1}, \\ \mathcal{P}_\omega X_{67,89} &\rightarrow -\gamma_3 \mathcal{P}_\omega X_{67,89} \gamma_3^{-1}. \end{aligned} \quad (19)$$

The transformations of A_{0123} and X_{45} are identical and produce fields in the adjoint representation of $U(N_c)$. The projection on $X_{67,89}$ produces fields in a two-index symmetric representation. More explicitly, for the gauge field the projection is

$$\mathcal{P}_\gamma \mathcal{P}_\omega A_\mu = \frac{1}{2} (\mathcal{P}_\omega A_\mu + J_{2N_c} \mathcal{P}_\omega A_\mu J_{2N_c}^{-1}). \quad (20)$$

The resulting theory is a $\mathcal{N} = 2$ $U(N_c)$ theory with a symmetric hypermultiplet. If one considers the orientifold projection alone, the theory is projected to $\mathcal{N} = 2$ $SO(2N_c)$ super Yang-Mills with a hypermultiplet in the two-index representation, we can think of this theory as the analog of the $SO(2N_c)$ gauge theory of the QCD case.

The D3/D7 spectrum is initially described by two $2N_c \times 2N_f$ chiral multiplets H^A describing strings from D3 to D7 branes and the reversed strings $\tilde{H}_A = \epsilon_{AB} H^{B\dagger}$. The orientifold and orbifold actions are as follows

$$H^A \rightarrow -i\epsilon_{AB} (\omega_3 H^B \omega_7^{-1})^*, \quad \mathcal{P}_\omega H^A \rightarrow \gamma_3 \mathcal{P}_\omega H^A \gamma_7^{-1}. \quad (21)$$

Therefore, the projections acting on flavor fields are

$$\mathcal{P}_\omega H^A = \frac{1}{2} (H^A + \epsilon_{AB} (H^B)^* J_{2N_f}^{-1}), \quad \mathcal{P}_\gamma \mathcal{P}_\omega H^A = \frac{1}{2} (\mathcal{P}_\omega H^A + J_{2N_c} \mathcal{P}_\omega H^A J_{2N_f}^{-1}). \quad (22)$$

The resulting massless field is a $\mathcal{N} = 2$ hypermultiplet in the $(N_c, \overline{N_f})$ representation, or N_f flavors in the fundamental representation of the $U(N_c)$ gauge group. In the theory obtained from the orientifold projection alone there are N_f hypermultiplets in the fundamental representation of the

$SO(2N_c)$ gauge group. Although the maximal possible flavor group is $U(2N_f)$, in the theory at hand it is actually reduced to $USp(2N_f)$, due to the coupling between the chiral components of the hypermultiplets with the chiral component of the vector multiplet in the superpotential

$$W \sim \tilde{H} X H. \quad (23)$$

Since X is in the adjoint of $SO(2N_c)$, flavor indices in the superpotential are contracted with an antisymmetric form, which is invariant under a $USp(2N_f) \subset U(2N_f)$ subgroup.

The massless spectrum of D7 branes, that describes the BPS sector of flavored operators, is split between vector fields in the 0123 and 6789 directions, A_{0123} and A_{6789} , and a scalar field in the 45 directions, X_{45} . Transformations act as

$$\begin{aligned} A_{0123} &\rightarrow -\omega_7 A_{0123}^T \omega_7^{-1}, & \mathcal{P}_\omega A_{0123} &\rightarrow \gamma_7 \mathcal{P}_\omega A_{0123} \gamma_7^{-1}, \\ X_{45} &\rightarrow -\omega_7 X_{45}^T \omega_7^{-1}, & \mathcal{P}_\omega X_{45} &\rightarrow \gamma_7 \mathcal{P}_\omega X_{45} \gamma_7^{-1}, \\ A_{6789} &\rightarrow -\omega_7 A_{6789}^T \omega_7^{-1}, & \mathcal{P}_\omega A_{6789} &\rightarrow -\gamma_7 \mathcal{P}_\omega A_{6789} \gamma_7^{-1}. \end{aligned} \quad (24)$$

Since the 8d Poincaré invariance is broken in the worldvolume of the D7 branes, the projection will be different for modes with dependence on the 6789 directions. The action (24) for A_{0123} and X_{45} is valid for parity even modes while the action for A_{6789} is valid for parity odd modes. This agrees with the A_{0123} and X_{45} components being scalar in the 6789 directions and A_{6789} being a vector component. The flavor group is $U(N_f)$, but the spectrum of BPS operators is different from the original theory since the hypermultiplet in the D3 sector is in the two-index symmetric representation and not in the adjoint.

The holographic dual description is type IIB string theory on $AdS_5 \times \mathbf{RP}^5$, with D7 probe branes that sit on top of O7 planes wrapping a $\mathbf{RP}^3 \subset \mathbf{RP}^5$ cycle. The $AdS_5 \times \mathbf{RP}^5$ geometry can be understood using a different basis of transformations. The O7 action is $\Omega_7 = \Omega_{R_{45}}(-1)^{F_L}$, while the \mathbb{Z}_2 singularity acts as a R_{6789} reflection on the geometry. Since O7 planes and O3 planes have the same effect on Ramond forms (cf. [29]), the combined action is equivalent to the action of an O3 plane $\Omega_3 = R_{6789}\Omega_7 = \Omega_{R_{456789}}(-1)^{F_L}$. The action of the O3 plane on $AdS_5 \times S^5$ is known to give the \mathbf{RP}^5 geometry, since it acts as a reflection on the space transverse to the D3 branes [30]. From the T-dual perspective this geometry without the O7 orientifold can be constructed from a stack of D4 branes sitting on $O4^-$ or $O4^+$, giving holographic duals with orthogonal or symplectic gauge groups.

3.2 From isospin to baryon chemical potential

The dynamics of the probe D7 branes in the D3 background are determined by the DBI action,

$$S_{DBI} = -T_7 \int d^8 \xi \, \text{Tr} \sqrt{-\det(G + 2\pi\alpha' F)}, \quad (25)$$

where ξ are the world-volume coordinates, G is the pull-back of the spacetime metric to the world volume and F is the field strength of the gauge fields on the brane. The $2N_f$ D7 branes in the $U(2N_c)$ theory are wrapping an $S^3 \subset S^5$. Writing the $AdS_5 \times S^5$ metric as

$$ds^2 = \frac{|y|^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{|y|^2} \sum_{i=4}^9 dy_i^2, \quad (26)$$

the D7's are localized at $y_8 = y_9 = 0$ and extend along all the other directions. As we have explained, the full projection identifies points in the geometry that map to each other under a reflection $y^i \rightarrow -y^i$.

An isospin chemical potential in the $U(2N_c)$ field theory is described by a background gauge field on the D7 brane

$$A_0 = i\mu J_{2N_f}. \quad (27)$$

More generally, this will be taken as the boundary condition for A_0 . Notice that this configuration survives both projections (24). To highlight the effect of the projection let us write the gauge potential on the D7 brane as

$$A_0 = \begin{pmatrix} H & C \\ C^\dagger & H' \end{pmatrix}, \quad (28)$$

where H, H' and C are $N_f \times N_f$ Hermitian and general complex matrices, respectively. The orientifold projection in (24) implies the conditions

$$H' = -H^T, \quad C = C^T, \quad (29)$$

so A_0 is in the adjoint of a $USp(2N_f)$ group,

$$J_{2N_f} A_0 + A_0^T J_{2N_f} = 0. \quad (30)$$

The orbifold projection based on the transformation (24) imposes the conditions

$$H = -H^T, \quad C = -C^*. \quad (31)$$

So H is reduced to a purely imaginary antisymmetric matrix and C to a purely imaginary symmetric matrix. The combination $\tilde{A}_0 = -iC - H$ belongs to the adjoint representation of a $U(N_f)$ group. One can check this by doing a global transformation $A_0 \rightarrow U A_0 U^\dagger$ with elements of the unbroken gauge group

$$U \omega_7 U^T = \omega_7, \quad U \gamma_7 U^\dagger = \gamma_7. \quad (32)$$

In terms of the $U(N_f)$ gauge field, the configuration (27) maps to

$$\tilde{A}_0 = \mu \mathbf{1}_{N_f}, \quad (33)$$

which corresponds to a baryon chemical potential. This shows that indeed the isospin chemical potential is projected to a baryon chemical potential. Notice that after the orientifold projection, because the gauge group $SO(2N_c)$ is real, the isospin chemical potential is equivalent to the ‘‘baryon number’’ chemical potential.

3.3 Validity of the equivalence

The definition of the non-Abelian DBI action without derivatives is ambiguous, as it is possible to use $[D_\mu, D_\nu] F_{\alpha\beta} = [F_{\mu\nu}, F_{\alpha\beta}]$ to convert between derivatives and field strengths. Nevertheless, a possible approach is to define the non-Abelian DBI action by giving an ordering prescription for the trace when the gradients of the field strength are small [38]. In this regime, the DBI action

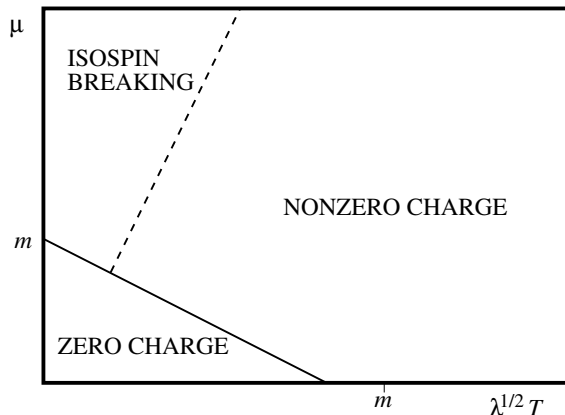


Figure 3: Sketch of the phase diagram in the chemical potential μ and temperature T plane, for either baryon or isospin chemical potential. The phase with spontaneous isospin breaking only exists for the case of an isospin chemical potential. The large N_c equivalence relating baryon and isospin chemical potential is valid in the confined and deconfined phases, but not in the region where isospin can be spontaneously broken.

does not change under the projections. We show this more explicitly in Appendix B. Under this assumption, we obtain the same results in the original and in the daughter theories, as long as all the components that are not invariant under the projection are zero. Notice that if restricted to questions about the ground state configuration with or without charge density, there are no ordering ambiguities; the configuration is Abelian so the separation between derivatives and field strengths is well-defined and any ordering prescription leads to the same results.

From the perspective of the dual field theory, this means that the parent and the daughter theories are equivalent in the large- N_c limit. In other words, the equivalence between the original and the daughter theory holds as long as the solution to the equations of motion obtained from the DBI action and the boundary conditions are invariant under the projection symmetry. We can fix the boundary conditions, but there is still the possibility that the correct solution breaks spontaneously one of the symmetries we have used to define the projection. This is a dynamical question, but fortunately one that can be answered in this context.

The phase diagram of D7 branes in the D3 background geometry was studied in ref. [31] for a baryon chemical potential, and also in refs. [32, 36, 33] for an isospin chemical potential.¹¹ We have sketched the phase diagram in Figure 3. For non-zero quark mass m there is a phase with no charge density, where quarks can form bound states of mass $\sim m/\sqrt{\lambda}$ and there is a discrete spectrum of low-spin mesonic states. For large enough chemical potential and/or temperature there is a phase transition,¹² and a finite charge density (baryon or isospin) appears. In this phase the quarks do not form bound states and the spectrum is continuous. The curves of the phase transition in the $(\mu_{B,I}, T)$ plane are the same for both baryon and isospin chemical potential, as

¹¹Notice that we are considering a baryon chemical potential in a theory where two of the adjoint fields have been changed to two-index symmetric representations. However, both theories are equivalent in the large- N_c limit. In the gravity side we have geometries with the same metric but different topologies, so solutions to the classical equations of motion are the same in both cases.

¹²The phase transition is first order at zero chemical potential, but changes to 3rd order at a tricritical point [34] at finite T and μ . At zero temperature the transition becomes second order [35].

we expect from the equivalence. The equivalence also holds in the finite density phase,¹³ but for an isospin chemical potential there is also a symmetry broken phase where vector mesons charged under the $U(1)$ isospin symmetry condense [36, 37]. The broken phase appears for values of the chemical potential larger than the scales given by the temperature and the mass of the mesons, which is proportional to the quark mass in these models. This implies in particular that at zero temperature and in the chiral limit the equivalence will fail. However, for massive quarks and small chemical potentials the equivalence will still be valid.

3.4 N_f/N_c corrections

It is interesting to study the question of N_f/N_c corrections in the theories with holographic duals and compare with QCD-like theories. Up to now, we have been working in a probe approximation for the flavor branes. From the field theory perspective this is equivalent to working in the 't Hooft limit of $N_f/N_c \rightarrow 0$, with no quark loops in planar diagrams. To find the effect of N_f/N_c corrections, one should compute the backreaction of the brane on the geometry. In principle, orientifold planes also produce a backreaction on the geometry of the same order; from the point of view of the field theory dual this corresponds to $1/N_c$ corrections not associated with flavor. However, in the limit where the number of flavor branes is large, but still much smaller than the number of color branes, $N_c \gg N_f \gg 1$, one can neglect the backreaction of the orientifold to leading order. From the field theory perspective, one can do a double expansion in N_f/N_c and $1/N_c$, where N_f/N_c corrections appear in planar diagrams with any number of flavor loops and $1/N_c$ corrections correspond to non-planar diagrams.

The comparison of N_f/N_c corrections is a bit subtle. On the one hand, the DBI action is the same for both the normal configuration dual to isospin chemical potential, and the orientifolded configuration dual to baryon chemical potential. Wess-Zumino terms with an even number of field strengths are also the identical. However, terms with an odd number of field strengths can differ. Consider, for instance, the coupling of the D7 to the RR potential C_6 ,

$$S_{WZ} = \mu_7 \int_{D7} C_6 \wedge \text{tr}(F). \quad (34)$$

Before introducing the orientifold, $\text{tr}(F) = 0$, since the gauge field on the brane is non-Abelian (27). The projection does not affect to the value of the trace, but for a D7 brane with an Abelian configuration (33), $\text{tr}(F) \neq 0$ in general, so this term is different in both cases and the backreaction of the D7 branes will also be different. Therefore, N_f/N_c corrections in the parent and daughter theories will be different in general and the equivalence only holds in the strict 't Hooft limit.

From the field theory perspective, we saw in section (2.3) that the condition (16) is not satisfied in the projection to a baryon chemical potential. For the supersymmetric theory we have to check two projections: both the orbifold projection, and the orientifold projection. The orbifold projection \mathcal{P}_γ mapping $SO(2N_c) \rightarrow U(N_c)$, whose action is given eqs. (20) and (22), naïvely seems to be regular. However, the projection is written in a basis where fundamental fields are in a real representation, in particular it is valid for Majorana fermions, while in the daughter theory fermions will be Dirac. In terms of a complex representation with Dirac fermions

¹³In ref. [32] it was observed that there is a 4-fold symmetry in the (μ_B, μ_I) plane.

one recovers the same kind of projection as in (3), which is not regular. Therefore, we also find from the field theory side that the equivalence will not hold for N_f/N_c corrections.

One can show indirectly that the orientifold projection \mathcal{P}_ω mapping $U(2N_c) \rightarrow SO(2N_c)$ in eqs. (18) and (22) is regular. For this, notice that there is a regular projection from $SO(4N_c) \rightarrow U(2N_c)$ with isospin chemical potential using a \mathbb{Z}_4 subgroup. This projection can be extended to $SO(4N_c) \rightarrow SO(2N_c)$ using a D_4 dihedral subgroup, with $\mathbb{Z}_4 \subset D_4$. Indeed, using the orbifold action $A_\mu \rightarrow J_{4N_c} A_\mu J_{4N_c}^{-1}$ to project $SO(4N_c) \rightarrow U(2N_c)$ and

$$L_{4N_c} = \begin{pmatrix} \mathbf{1}_{2N_c} & 0 \\ 0 & -\mathbf{1}_{2N_c} \end{pmatrix}, \quad (35)$$

to further project to $SO(2N_c)$ through the action $A_\mu \rightarrow L_{4N_c} A_\mu L_{4N_c}^{-1}$, one generates a regular representation of D_4 .¹⁴ We can then group flavor fields in a $4N_c \times 4N_f$ real matrix

$$\mathcal{H} = \begin{pmatrix} H_1 + H_1^* & H_2 + H_2^* \\ i(H_1^* - H_1) & i(H_2^* - H_2) \end{pmatrix}, \quad (36)$$

and perform the orientifold projection (22) in this basis as

$$\mathcal{P}_\omega \mathcal{H} = \frac{1}{2} \left(\mathcal{H} + L_{4N_c} \mathcal{H} \Omega_{4N_f}^{-1} \right), \quad (37)$$

where

$$\Omega_{4N_f}^{-1} = \begin{pmatrix} 0 & -J_{2N_f}^{-1} \\ J_{2N_f}^{-1} & 0 \end{pmatrix}. \quad (38)$$

Since $\text{tr} \Omega_{4N_f} = 0$ and $\Omega_{4N_f}^2 = \mathbf{1}_{4N_f}$, this shows that the orientifold projection in the flavor sector is also regular. Notice that the projection acting on fields on the flavor D7 branes is determined by Ω_{4N_f} , so from the perspective of the field theory on the D7 branes this is a regular projection. In the holographic duals to both the parent $U(2N_c)$ and daughter $SO(2N_c)$ theories the gauge field configuration on the D7 brane is traceless, so the issue of terms with an odd number of field strengths does not arise in this case.

4 Conclusion

We have used a holographic construction to demonstrate a large- N_c equivalence between theories with baryon and isospin chemical potential in the 't Hooft limit. The equivalence is valid in the region of the phase diagram where neither isospin nor baryon symmetry are spontaneously broken. The allowed region contains a small temperature/chemical potential phase with no charge density and mesonic bound states, and a large temperature/chemical potential phase with a finite charge density and a continuous spectrum. Although the charge density vanishes in the low temperature phase in the classical supergravity approximation, it will be nonzero when one takes Hawking radiation from the black hole into account. From the perspective of the dual field theory, this indicates that any charge density is suppressed in the large- N_c limit, compared to the charge density that is present in the high temperature phase. In other words, in the low temperature phase the equivalence we have presented relates the leading order large- N_c behavior of “vacuum”

¹⁴The elements of the group are $\{\mathbf{1}_{4N_c}, -\mathbf{1}_{4N_c}, J_{4N_c}, -J_{4N_c}, L_{4N_c}, -L_{4N_c}, J_{4N_c} L_{4N_c}, L_{4N_c} J_{4N_c}\}$.

thermodynamic properties of the theory with either baryon or isospin chemical potential, but it does not provide information about other details, such as properties of the thermal gas of mesons and baryons, which are $1/N_c$ suppressed. The large N_c equivalence does relate non-trivial dependence on temperature and chemical potential in the deconfined phase (with unbroken isospin and baryon number), where there is temperature and chemical potential dependence at leading order in N_c .

Accepting the validity of gauge/string duality, our analysis suggests that large N_c equivalences relating theories with differing chemical potentials may be valid more generally (when appropriate symmetry realizations hold). Exactly the same projections which we have used for supersymmetric theories can also be used to relate $U(2N_c)$ QCD with $2N_f$ flavors and isospin chemical potential to $U(N_c)$ QCD with N_f flavors and baryon chemical potential. However, because there is no known gravitational dual of QCD, different methods are needed to construct a purely field theoretic proof which applies to this case. We hope to revisit these points in the future.

Acknowledgments

We would like to thank Johanna Erdmenger and Patrick Kerner for stimulating discussions and comments. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-96ER40956. The work of M.H. is supported by Japan Society for the Promotion of Science Postdoctoral Fellowship for Research Abroad.

A A quick introduction to the sign problem

Consider pure Yang-Mills theory. In lattice Monte-Carlo simulations, one generates field configurations with probability weight $e^{-S_{YM}}/Z_{YM}$ (times Haar measure), where S_{YM} is the Euclidean action and the partition function $Z_{YM} \equiv \int dA_\mu e^{-S_{YM}[A]}$. Expectation values are approximated by taking the average over many configurations generated by a Markov chain:

$$\langle \mathcal{O} \rangle \equiv \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \mathcal{O}[A_\mu^{(i)}] = \frac{1}{Z_{YM}} \int dA_\mu \mathcal{O}[A] e^{-S_{YM}[A]}. \quad (39)$$

Here k is the number of lattice field configurations generated in the simulation and $i = 1, \dots, k$ is a label distinguishing them. In the sequence of configurations, more likely configurations appear more often; this is known as the “importance sampling”.

The key assumption, which is valid for pure gauge theories, is that the weight $e^{-S_{YM}}/Z_{YM}$ is *real and positive*, so that the integrand of the functional integral may be regarded as a probability measure. This condition can be broken when there are fermions in the theory.

To deal with fermions in a lattice theory, one performs the Grassmann integral by hand. For example, the expectation value of the chiral condensate $\bar{\psi}\psi$ can be expressed as

$$\langle \bar{\psi}\psi \rangle = \frac{\int dA_\mu \text{Tr} \mathcal{D}^{-1}[A] \cdot \det \mathcal{D}[A] \cdot e^{-S_{YM}[A]}}{\int dA_\mu \det \mathcal{D}[A] \cdot e^{-S_{YM}[A]}}, \quad (40)$$

where \mathcal{D} denotes the lattice Dirac operator. If $\det \mathcal{D}[A]$ is real and positive for any gauge field A_μ , then one may simulate this system by using the effective action $S_{eff}[A] = S_{YM}[A] - \log \det \mathcal{D}[A]$.

However, the determinant $\det \mathbb{D}[A]$ can, in general, be complex, and then standard Monte-Carlo techniques cannot be applied. This is the so-called “sign problem” (or more properly “phase problem”), the word “sign” referring to a possible negative sign of the determinant. QCD with baryon chemical potential suffers from the sign problem, while QCD with isospin chemical potential and the $SO(2N_c)$ gauge theory with fermion chemical potential are “sign-free”.

One standard approach for dealing with the sign problem is the so-called “reweighing” method. Consider the *phase quenched* ensemble with weight $|\det \mathbb{D}|e^{-S_{YM}}$. When the number of flavors is even, the phase-quenched version of QCD with baryon chemical potential is identical to QCD with an isospin chemical potential. If $\langle \cdots \rangle_B$ denotes expectations in QCD with a baryon chemical potential, and $\langle \cdots \rangle_I$ expectations with an isospin potential, then it is immediate that

$$\langle \mathcal{O} \rangle_B = \frac{\langle \mathcal{O} \cdot e^{i\eta} \rangle_I}{\langle e^{i\eta} \rangle_I}, \quad (41)$$

where $e^{i\eta}$ is the phase of the fermion determinant in the presence of a baryon chemical potential, $e^{i\eta} \equiv \det \mathbb{D} / |\det \mathbb{D}|$. Because one can apply standard Monte-Carlo simulation techniques to the phase-quenched ensemble, one can in principle evaluate the expectation value in the full theory by computing both numerator and denominator in the identity (41); this is the reweighing method. However, in practice reweighing works only when the phase does not fluctuate violently. Phase fluctuations grow as the chemical potential μ_B is increased, and as the lattice volume V grows. The logarithm of the fermion determinant is extensive, $\ln \det \mathbb{D} = O(V)$, and (with a non-zero baryon chemical potential), so is its imaginary part, η . This implies that both numerator and denominator of eq. (41) vanish exponentially in the thermodynamic limit, making their estimation via Monte Carlo methods increasingly problematic as the volume $V \rightarrow \infty$.

B Projection of the DBI action

The DBI action of a $D(d-1)$ brane may be expressed as

$$S_{DBI} = -T_{d-1} \int d^d \xi \quad \mathbf{S}' \mathbf{Tr}_{2N_f} \sqrt{-\det (G + 2\pi\alpha' F)}, \quad (42)$$

where we use the notation $\mathbf{S}' \mathbf{Tr}_{2N_f}$ for a trace of $2N_f \times 2N_f$ matrices with a predetermined ordering prescription that is, however, unknown. Up to F^4 terms, it should coincide with the symmetrized trace prescription [38].

For a d -dimensional brane, one can write the determinant as

$$\det (G + 2\pi\alpha' F) = \frac{1}{d!} \epsilon^{\mu_1 \cdots \mu_d} \epsilon^{\nu_1 \cdots \nu_d} (G_{\mu_1 \nu_1} + 2\pi\alpha' F_{\mu_1 \nu_1}) \cdots (G_{\mu_d \nu_d} + 2\pi\alpha' F_{\mu_d \nu_d}), \quad (43)$$

where $G_{\mu_1 \nu_1}$ is the pullback of the metric and is proportional to the identity matrix. The field strengths $F_{\mu_1 \nu_1}$ are proportional to the generators of the gauge group on the brane. Since this is a matrix product one has to define the order, we will not assume a particular ordering in the following. We can extract the metric factors as

$$\begin{aligned} \det (G + 2\pi\alpha' F) &= \frac{1}{d!} \epsilon^{\mu_1 \cdots \mu_d} \epsilon^{\nu_1 \cdots \nu_d} G_{\mu_1 \alpha_1} \cdots G_{\mu_d \alpha_d} (\delta_{\nu_1}^{\alpha_1} + 2\pi\alpha' G^{\alpha_1 \beta_1} F_{\beta_1 \nu_1}) \cdots (\delta_{\nu_d}^{\alpha_d} + 2\pi\alpha' G^{\alpha_d \beta_d} F_{\beta_d \nu_d}) \end{aligned}$$

$$= \frac{1}{d!} (\det G) \epsilon^{\mu_1 \dots \mu_d} \epsilon_{\alpha_1 \dots \alpha_d} (\delta_{\nu_1}^{\alpha_1} + 2\pi\alpha' G^{\alpha_1 \beta_1} F_{\beta_1 \nu_1}) \dots (\delta_{\nu_d}^{\alpha_d} + 2\pi\alpha' G^{\alpha_d \beta_d} F_{\beta_d \nu_d}). \quad (44)$$

Alternatively, one can write the determinant as

$$\begin{aligned} \det(G + 2\pi\alpha' F) &= \frac{1}{d!} \epsilon^{\mu_1 \dots \mu_d} \epsilon^{\nu_1 \dots \nu_d} G_{\alpha_1 \nu_1} \dots G_{\alpha_d \nu_d} (\delta_{\mu_1}^{\alpha_1} + 2\pi\alpha' G^{\alpha_1 \beta_1} F_{\beta_1 \mu_1}) \dots (\delta_{\mu_d}^{\alpha_d} + 2\pi\alpha' G^{\alpha_d \beta_d} F_{\beta_d \mu_d}) \\ &= \frac{1}{d!} (\det G) \epsilon^{\nu_1 \dots \nu_d} \epsilon_{\alpha_1 \dots \alpha_d} (\delta_{\mu_1}^{\alpha_1} + 2\pi\alpha' G^{\alpha_1 \beta_1} F_{\beta_1 \mu_1}) \dots (\delta_{\mu_d}^{\alpha_d} + 2\pi\alpha' G^{\alpha_d \beta_d} F_{\beta_d \mu_d}). \end{aligned} \quad (45)$$

Since $G^{\alpha\beta} F_{\beta\nu} = -G^{\alpha\beta} F_{\nu\beta}$, after relabeling indices $\nu_i \leftrightarrow \mu_i$ in (45) and comparing with (44), one finds that $\det(G + 2\pi\alpha' F) = \det(G - 2\pi\alpha' F)$. Therefore, only even powers of the field strength F appear in the expansion of the determinant, which can be seen as a manifestation of charge conjugation invariance. Taking the square root and defining $\sigma_N(T^N)$ as a possible ordering of N generators T appearing in the trace,¹⁵ we have

$$\begin{aligned} \mathbf{S}' \text{Tr}_{2N_f} \sqrt{-\det(G + 2\pi\alpha' F)} &= \sqrt{-\det G} \\ &\times \left[2N_f + \sum_{N \geq 1} (\alpha')^{2N} \sum_{\sigma_{2N}} \sum_{k \geq 1} \left(\prod_{q=1}^k \sum_{n_q=0}^{[d/2]} \right) \delta_{\sum_q n_q = N} c_{n_1, n_2, \dots, n_k}^{N, k} \text{tr} \sigma_{2N} (F^{2n_1} F^{2n_2} \dots F^{2n_k}) \right]. \end{aligned} \quad (46)$$

Where we have suppressed spacetime indices and denote F^n as a product of n field strengths appearing in the determinant. The largest possible power is d if d is even or $d-1$ if d is odd. The action has to be Hermitian, this implies that given some ordering σ_{2N} , the reversed ordering σ_{2N}^T also appears with the same coefficients. For instance, if we have $\text{tr}(F_1 F_2 \dots F_{n-1} F_n)$, the Hermitian conjugate is $\text{tr}(F_n F_{n-1} \dots F_2 F_1)$.

Although expression (46) takes the form of an α' expansion, we are overlooking other α' corrections involving derivatives of the field strength and α' corrections that depend on the background metric. The former can formally be included by allowing DF factors inside the traces; many new terms (with unknown coefficients) would appear but the basic structure would not change if discrete symmetries (C,P,T) are not broken. We will just assume that gradients of the field strength are much smaller than their magnitude. Corrections to the geometry are more problematic, as α' corrections are, in general, different in the presence of an orientifold plane. But since we are both in the supergravity approximation and in a probe limit, we can neglect these corrections so the coefficients of the expansion are not affected. Under these assumptions the DBI action of the daughter theory is the naïve projection of the original theory, as we will now show.

Let us write the field strength of the gauge field on the D-brane as

$$F = \begin{pmatrix} H & C \\ C^\dagger & H' \end{pmatrix}. \quad (47)$$

Here and in the following, we suppress spacetime indices. Under a global $U(2N_f)$ transformation taking $F \rightarrow PFP^{-1}$, with P defined in (6), the field strength transforms to

$$PFP^{-1} = \frac{1}{2} \begin{pmatrix} H + H' + i(C^\dagger - C) & H - H' + i(C^\dagger + C) \\ H - H' - i(C + C^\dagger) & H + H' - i(C^\dagger - C) \end{pmatrix}. \quad (48)$$

¹⁵Since σ_N is defined for a trace, it maps to an element of the quotient of the permutation group over the cyclic group S_n/C_n , which corresponds to the conjugacy classes of S_n . These can be classified using Young tableaux.

After applying the conditions (29) and (31) which follow from the projection, the transformed field strength is block-diagonal,

$$PFP^{-1} = \begin{pmatrix} H - iC & 0 \\ 0 & H + iC \end{pmatrix} = \begin{pmatrix} \tilde{F}^T & 0 \\ 0 & -\tilde{F} \end{pmatrix}. \quad (49)$$

Here \tilde{F} is the $U(N_f)$ gauge field of the daughter theory. Any power of the field strength F^n has the same block-diagonal form after the global transformation, so for $\sum_k \ell_k = L$ factors, the trace is

$$\begin{aligned} \text{tr } \sigma_L \left(\prod_k F^{\ell_k} \right) &= \text{tr } \sigma_L \left(P \prod_k F^{\ell_k} P^{-1} \right) = \text{tr } \sigma_L \left(\prod_k (\tilde{F}^T)^{\ell_k} \right) + \text{tr } \sigma_L \left(\prod_k (-\tilde{F})^{\ell_k} \right) \\ &= \text{tr } \sigma_L^T \left(\prod_k \tilde{F}^{\ell_k} \right) + (-1)^L \text{tr } \sigma_L \left(\prod_k \tilde{F}^{\ell_k} \right). \end{aligned} \quad (50)$$

Since L is always even in the expansion (46), the phase factor is trivial $(-1)^L = (-1)^{2N} = 1$. For each ordering σ_{2N} in (46) we get two terms, one corresponding to the same ordering for the $U(N_f)$ gauge field and another one corresponding to the reversed order. From the Hermiticity of the action in the $U(2N_f)$ theory, we should have another contribution in the projected action that is exactly the same but whose origin is a term with reversed order σ_{2N}^T . Adding the two together, we have a Hermitian action where the traces are projected to

$$\text{tr } \sigma_{2N} (F^{2n_1} F^{2n_2} \dots F^{2n_k}) \rightarrow 2 \text{tr } \sigma_{2N} (\tilde{F}^{2n_1} \tilde{F}^{2n_2} \dots \tilde{F}^{2n_k}). \quad (51)$$

Finally, using expression (46),

$$\mathbf{S}' \text{Tr}_{2N_f} \sqrt{-\det (G + 2\pi\alpha' F)} \rightarrow 2 \mathbf{S}' \text{Tr}_{N_f} \sqrt{-\det (G + 2\pi\alpha' \tilde{F})} \quad (52)$$

This proves the equivalence of the ordering prescription in the original and the projected actions. Comparing with the action (42), the tension of the D-brane in the $U(2N_f)$ theory is half the tension of the $U(N_f)$ theory. However, this is compensated by the volume of the internal space, that is halved when we project from the S^5 to the \mathbf{RP}^5 geometry.

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